

Fourier Analysis and q -Gaussian Functions: Analytical and Numerical Results

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Abstract. *It is a consensus in signal processing that the Gaussian kernel and its partial derivatives enable the development of robust algorithms for feature detection. Fourier analysis and convolution theory have central role in such development. In this paper we collect theoretical elements to follow this avenue but using the q -Gaussian kernel that is a nonextensive generalization of the Gaussian one. Firstly, we review some theoretical elements behind the one-dimensional q -Gaussian and its Fourier transform. Then, we consider the two-dimensional q -Gaussian and we highlight the issues behind its analytical Fourier transform computation. We analyze the q -Gaussian kernel in the space and Fourier domains using the concepts of space window, cut-off frequency, and the Heisenberg inequality.*

1. Introduction

Feature extraction is an essential step for image analysis and computer vision tasks such as image matching and object recognition [Jain 1989]. The specific case of edge detection, has been extensively considered in the image processing literature [Gonzalez 1992]. In this subject, the Gaussian kernel and its partial derivatives have inspired a wide range of works in the image analysis literature for the development of multiscale approaches [Lopez-Molina et al. 2013].

These works had established the background for multiscale representation based on the viewpoint of the functional structure of digital images [Florack et al. 1992, Koenderink 1984]. Basically, the grayscale of the observed image is realized as a general function f of the space of square integrable functions on \mathbb{R}^2 , denoted by $L^2(\mathbb{R}^2)$. The linear scale-space is generated by the convolution with scaled Gaussian kernels with filtering properties analyzed in the frequency space given by the Fourier transform. Such approach can be seen from the isotropic diffusion equation viewpoint, which opens the possibility of generating more general multiscale representations based on the anisotropic diffusion equation [Perona and Malik 1990].

The Gaussian kernel plays also a fundamental role in statistical physics due to the fact that an enormous amount of phenomena in nature follow the Gaussian distribution and the extensive thermostatics governed by the Boltzmann-Gibbs entropy and the standard central limit theorem [Liboff 1990]. More recently, Tsallis nonextensive entropy and

generalizations of the central limit theorem gives the foundations for noextensive counterparts of the Boltzmann-Gibbs statistical mechanics [Tsallis 2009, Tsallis 1988]. In this context, generalization of Gaussian smoothing kernel within the Tsallis nonextensive scenario, named q -Gaussian, has been proposed [Tsallis et al. 1995]. Also, the difference of q -Gaussian kernels is introduced in [Assirati et al. 2014] to build a method for edge extraction in digital images. These works demonstrate the potential of the q -Gaussian based methods by comparing the obtained results with the ones obtained with traditional techniques that rely on the Gaussian function [Gallao and Rodrigues 2015]. However, the behaviour of the q -Gaussian in the frequency domain has been ignored.

In this paper, we collect theoretical elements, published in the references [Borges et al. 2004, Borges 1998, Daz and Pariguan 2009], to perform such analysis. We review the q -exponential function and the q -Gaussian distribution. Then, we offer details of the Fourier transform computation for $d = 1$. The two-dimensional q -Gaussian is also considered from the analytical and numerical viewpoint. In the experimental results, we analyze the q -Gaussian in the frequency domain and compare its profile with the Gaussian one. In fact, we consider the Fourier transform of the one-dimensional q -Gaussian which emphasizes the fact that a q -Gaussian is a low-pass filter. We study the size of the space window and analyze the influence of the parameter q in the cut-off frequency and in the Heisenberg inequality.

The paper is organized as follows. Section 2 focuses on Tsallis entropy, the q -exponential and q -Gaussian and summarizes some of their basic properties. The Fourier transform computation of the q -Gaussian is discussed on sections 3 and 4. The computational results are presented on section 6. The Appendices A-F complete the material with details about the q -Gaussian and its Fourier transform as well as the special functions Gamma, Whittaker, Bessel and Beta functions, used along the text.

2. Tsallis Entropy and q -Gaussian

In the last decade, Tsallis [Tsallis 1988] has proposed the following generalized nonextensive entropic form:

$$S_q = k \frac{1 - \sum_{i=1}^W p_i^q}{q - 1}, \quad (1)$$

where k is a positive constant, p_i is a probability distribution and $q \in \mathbb{R}$ is called the entropic index. This expression recovers the Shannon entropy in the limit $q \rightarrow 1$. The Tsallis entropy offers a new formalism in which the real parameter q quantifies the level of nonextensivity of a physical systems [Tsallis 1999]. In particular a general principle of maximum entropy (PME) has been considered to find out the distribution p_i to describe such systems. In this PME, the goal is to find the maximum of S_q subjected to:

$$\sum_{i=1}^W p_i = 1, \quad (2)$$

$$\frac{\sum_{i=1}^W e_i p_i^q}{\sum_{i=1}^W p_i^q} = U_q, \quad (3)$$

where U_q is a known application dependent value and e_i represent the possible states of the system (in image processing, the gray-level intensities). Expression (2) just imposes

that p_i is probability and equation (3) is a generalized expectation value of the e_i (if $q = 1$ we get the usual mean value). The proposed PME can be solved using Lagrange multipliers and the solution has the form [Tsallis 1999, Tsallis et al. 1998]:

$$p_j = \frac{\left[1 - (1 - q)\tilde{\beta}e_j\right]^{\frac{1}{1-q}}}{\tilde{Z}_q}, \quad (4)$$

where $\tilde{\beta}$ and \tilde{Z}_q are defined by the expressions:

$$\tilde{\beta} = \frac{\beta}{\sum_{j=1}^W p_j^q + (1 - q)\beta U_q},$$

$$\tilde{Z}_q = \sum_{j=1}^W \left[1 - (1 - q)\tilde{\beta}e_j\right]^{\frac{1}{1-q}},$$

with β being the Lagrange multiplier associated with the constraint given by expression (3) and, if $q < 1$, then $p_i = 0$ whenever $1 - (1 - q)\tilde{\beta}e_j < 0$ (cut-off condition). The expressions (1) and (4) inspire the definition of the q -exponential function [Tsallis 1994]:

$$\exp_q(x) = \begin{cases} [1 + (1 - q)x]^{\frac{1}{1-q}}, & \text{if } 1 + (1 - q)x > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

It can be shown that the traditional exponential function (\exp) is given by the limit:

$$\exp(x) = \lim_{q \rightarrow 1} \exp_q(x). \quad (6)$$

The equation (6) motivates the definition of the d -dimensional q -Gaussian as [Daz and Pariguan 2009]:

$$G_{d,q}(\mathbf{x}, \Sigma, \beta) = C_{d,q}(\Sigma, \beta) \exp_q(-\beta \mathbf{x}^T \Sigma^{-1} \mathbf{x}), \quad (7)$$

where:

$$C_{d,q}(\Sigma, \beta) = \left(\int_{\mathbb{R}^d} \exp_q(-\beta \mathbf{x}^T \Sigma^{-1} \mathbf{x}) d\mathbf{x} \right)^{-1}. \quad (8)$$

and Σ is the covariance matrix (symmetric and positive definite). Due to expression (6) it is straightforward to show that:

$$\lim_{q \rightarrow 1} \exp_q(-\beta \mathbf{x}^T \Sigma^{-1} \mathbf{x}) = \exp(-\beta \mathbf{x}^T \Sigma^{-1} \mathbf{x}). \quad (9)$$

Consequently, depending on the $C_{d,q}(\beta)$ functional form, we can recover the d -dimensional Gaussian:

$$G_d(\mathbf{x}, \Sigma) = \left[(2\pi)^{d/2} |\Sigma|^{1/2} \right]^{-1} \exp \left(-\frac{1}{2} \mathbf{x}^T \Sigma^{-1} \mathbf{x} \right), \quad (10)$$

by taking the limit of $G_{d,q}(\mathbf{x}, \Sigma, \beta)$ at $q \rightarrow 1$. In the Appendices C and D we give details of that developments showing that, if $\beta = 1/2$, then we obtain the one and two-dimensional Gaussian functions in this way. The corresponding q -expressions are reproduced bellow.

a) One-Dimensional q -Gaussian (see Appendix C):

$$G_{1,q}(x, \sigma, \beta) = C_{1,q}(\sigma, \beta) \exp_q \left(-\frac{\beta}{\sigma^2} x^2 \right) = C_{1,q}(\beta) \left[1 + (q-1) \frac{\beta}{\sigma^2} x^2 \right]^{\frac{1}{1-q}}, \quad (11)$$

where:

$$C_{1,q}(\sigma, \beta) = \frac{\Gamma \left(\frac{1}{q-1} \right) \left[(q-1) \frac{\beta}{\sigma^2} \right]^{1/2}}{\sqrt{\pi} \Gamma \left(\frac{1}{q-1} - \frac{1}{2} \right)}, \quad 3 > q > 1, \quad (12)$$

$$C_{1,q}(\sigma, \beta) = \frac{\Gamma \left(\frac{1}{1-q} + \frac{3}{2} \right) \left((1-q) \frac{\beta}{\sigma^2} \right)^{1/2}}{\sqrt{\pi} \Gamma \left(\frac{1}{1-q} + 1 \right)}, \quad q < 1, \quad |\mathbf{x}| \leq \left((1-q) \frac{\beta}{\sigma^2} \right)^{-1/2}. \quad (13)$$

b) Two-Dimensional q -Gaussian (see Appendix D):

$$G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta(2-q)}{\pi \sqrt{|\Sigma|}} \left[1 + (1-q) (-\beta \mathbf{x}^T \Sigma^{-1} \mathbf{x}) \right]^{\frac{1}{1-q}}, \quad 1 < q < 2, \quad (14)$$

$$G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta(2-q)}{\pi \sqrt{|\Sigma|}} \left[1 + (1-q) (-\beta \mathbf{x}^T \Sigma^{-1} \mathbf{x}) \right]^{\frac{1}{1-q}}, \quad q < 1, \quad (15)$$

subject to the constraint:

$$0 < (\mathbf{x}^T \Sigma^{-1} \mathbf{x})^{1/2} < \frac{1}{\sqrt{\beta(1-q)}}. \quad (16)$$

3. Fourier Transform of 1D q -Gaussian

From expression (11) we can compute the Fourier transform of the one-dimensional q -Gaussian as:

$$\mathcal{F} \left(C_{1,q}(\sigma, \beta) \exp_q \left(-\frac{\beta}{\sigma^2} x^2 \right); y \right)$$

$$= C_{1,q}(\sigma, \beta) \mathcal{F} \left(\exp_q \left(-\frac{\beta}{\sigma^2} x^2 \right); y \right),$$

where $\mathcal{F}(g(x); y)$ means the Fourier transform of function g computed at the frequency y . In this paper, the Fourier transform is defined by:

$$\mathcal{F}(g(x); y) = \int_{-\infty}^{+\infty} \exp(-2j\pi xy) g(x) dx. \quad (17)$$

The Appendix E develops the computation of the Fourier transform for a q -exponential. So, using expressions (85) and (89) we obtain:

$$\begin{aligned} \mathcal{F}(G_{1,q}(x, \sigma, \beta); y) &= C_{1,q}(\sigma, \beta) \left[(q-1) \frac{\beta}{\sigma^2} \right]^{-1/2} \times \\ &\left(-\text{sign} \left(2\pi \left[(q-1) \frac{\beta}{\sigma^2} \right]^{-1/2} y \right) 2\pi \left(2^{\frac{1}{1-q}} \right) \frac{|2\pi [(q-1) \frac{\beta}{\sigma^2}]^{-1/2} y|^{\frac{1}{q-1}-1}}{\Gamma\left(\frac{1}{q-1}\right)} \right) \times \\ &W_{0, \frac{1}{2} + \frac{1}{1-q}} \left(2|2\pi [(q-1) \frac{\beta}{\sigma^2}]^{-1/2} y| \right), \quad 1 < q < 3, \end{aligned} \quad (18)$$

where $W_{0, \frac{1}{2} + \frac{1}{1-q}}$ are Whittaker functions, defined in the Appendix F. Consequently :

$$\begin{aligned} \text{abs} [\mathcal{F}(G_{1,q}(x, \sigma, \beta); y)] &= \text{abs} [C_{1,q}(\beta)] \times \\ &\text{abs} \left[\pi^{3/2} \frac{[(q-1)a]^{-1/2}}{\Gamma\left(\frac{1}{q-1}\right)} \left(i \exp \left(\frac{i\pi}{2} \left(-\frac{1}{2} - \frac{1}{1-q} \right) \right) \right) \right] \times \\ &\text{abs} \left[\left(\frac{z}{2} \right)^{\frac{1}{q-1}-\frac{1}{2}} \mathbb{J}_{-\frac{1}{2} + \frac{1}{q-1}}(iz) + i \frac{\left[\left(\frac{z}{2} \right)^{\frac{1}{q-1}-\frac{1}{2}} \mathbb{J}_{-\frac{1}{2} + \frac{1}{q-1}}(iz) \cos \left(\pi \left(-\frac{1}{2} + \frac{1}{q-1} \right) \right) - (i)^{\frac{1}{2}-\frac{1}{q-1}} S(z) \right]}{\sin \left(\pi \left(-\frac{1}{2} + \frac{1}{q-1} \right) \right)} \right], \end{aligned} \quad (19)$$

where $z = |2\pi [(q-1)a]^{-1/2} y|$, $a = \frac{\beta}{\sigma^2}$ and $1 < q < 3$ and:

$$S(z) = \sum_{k=0}^{+\infty} \frac{\left(\frac{z^2}{4} \right)^k}{k! \Gamma \left(\frac{1}{2} - \frac{1}{q-1} + k + 1 \right)}$$

Analogously, we can show that:

$$\mathcal{F}(G_{1,q}(x, \sigma, \beta); y) = C_{1,q}(\sigma, \beta) \frac{\sqrt{\pi}}{\left((1-q) \frac{\beta}{\sigma^2}\right)^{1/2}} \times$$

$$\left(-\frac{\left((1-q) \frac{\beta}{\sigma^2}\right)^{1/2}}{\pi y} \right)^{\frac{1}{1-q} + \frac{1}{2}} \Gamma\left(\frac{1}{1-q} + 1\right) \mathbb{J}_{\frac{1}{1-q} + \frac{1}{2}}\left(-\frac{2\pi y}{\left((1-q) \frac{\beta}{\sigma^2}\right)^{1/2}}\right), \quad q < 1, y \neq 0, \quad (20)$$

$$\mathcal{F}(G_{1,q}(x, \sigma, \beta); 0)$$

$$= C_{1,q}(\sigma, \beta) \frac{2^{\frac{2}{1-q}+1}}{\left((1-q) \frac{\beta}{\sigma^2}\right)^{1/2}} \frac{\Gamma\left(\frac{1}{(1-q)} + 1\right) \Gamma\left(\frac{1}{(1-q)} + 1\right)}{\Gamma\left(\frac{2}{1-q} + 2\right)}, \quad q < 1, y = 0, \quad (21)$$

with $\mathbb{J}_{\frac{1}{1-q} + \frac{1}{2}}$ being the Bessel functions (see Appendix F).

4. Fourier Transform of 2D q -Gaussian

In this section we offer a sketch of the Fourier transform for the two-dimensional q -Gaussian, defined by expressions (14)-(15), for a diagonal matrix $\Sigma = \text{diag}(\sigma_1^2, \sigma_2^2)$ which is computed as follows:

$$\mathcal{F}(G_{2,q}(\mathbf{x}, \Sigma, \beta); \omega_1, \omega_2) = C_{2,q} \times \quad (22)$$

$$= (\Sigma, \beta) \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[-2\pi j(\omega_1 x + \omega_2 y)] \left[1 + (1-q) \left(-\beta \left(\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2}\right)\right)\right]^{\frac{1}{1-q}} dx dy, \quad (23)$$

where $\mathbf{x} = (x, y)$. Therefore, we can re-write this expression as:

$$\mathcal{F}(G_{2,q}(\mathbf{z}, \Sigma, \beta); \omega_1, \omega_2) = C_{2,q}(\Sigma, \beta) \sqrt{|\Sigma|} \times \quad (24)$$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp[-2\pi j(\omega_1 z_1 + \omega_2 z_2)] \left[1 + (1-q) \left(-\beta (z_1^2 + z_2^2)\right)\right]^{\frac{1}{1-q}} dz_1 dz_2 \quad (25)$$

where $z_1 = x/\sigma_1$ and $z_2 = y/\sigma_2$.

Considering just the double integral of equation (25) we can write:

$$\int_{-\infty}^{+\infty} \exp[-2\pi j\omega_2 z_2] \left\{ \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_1 z_1] \left[1 + (1-q) \left(-\beta (z_1^2 + z_2^2)\right)\right]^{\frac{1}{1-q}} dz_1 \right\} dz_2$$

$$\begin{aligned}
&= \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_2 z_2] \left\{ \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_1 z_1] [1 + (q-1)(\beta z_1^2 + \beta z_2^2)]^{\frac{1}{1-q}} dz_1 \right\} dz_2 \\
&= \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_2 z_2] \left\{ \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_1 z_1] [1 + (q-1)\beta z_2^2 + (q-1)\beta z_1^2]^{\frac{1}{1-q}} dz_1 \right\} dz_2 \\
&= \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_2 z_2] \times \\
&\quad \left\{ \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_1 z_1] [1 + (q-1)\beta z_2^2]^{\frac{1}{1-q}} \left[1 + \left(\frac{q-1}{1 + (q-1)\beta z_2^2} \right) \beta z_1^2 \right]^{\frac{1}{1-q}} dz_1 \right\} dz_2 \\
&= \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_2 z_2] [1 + (q-1)\beta z_2^2]^{\frac{1}{1-q}} \times \\
&\quad \left\{ \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_1 z_1] \left[1 + (q-1) \left(\frac{\beta}{1 + (q-1)\beta z_2^2} \right) z_1^2 \right]^{\frac{1}{1-q}} dz_1 \right\} dz_2 \\
&= \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_2 z_2] [1 + (q-1)\beta z_2^2]^{\frac{1}{1-q}} \times \\
&\quad \left\{ \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_1 z_1] [1 + (q-1)\zeta z_1^2]^{\frac{1}{1-q}} dz_1 \right\} dz_2,
\end{aligned}$$

where:

$$\zeta = \left(\frac{\beta}{1 + (q-1)\beta z_2^2} \right).$$

Let

$$F_1(\omega_1, \zeta(z_2, q), q) = \int_{-\infty}^{+\infty} \exp[-2\pi j\omega_1 z_1] [1 + (q-1)\zeta z_1^2]^{\frac{1}{1-q}} dz_1.$$

Therefore, we can return to expression (25) and write:

$$\mathcal{F}(G_{2,q}(\mathbf{z}, \Sigma, \beta); \omega_1, \omega_2) = C_{2,q}(\Sigma, \beta) \sqrt{|\Sigma|} \times \quad (26)$$

$$\int_{-\infty}^{+\infty} \exp[-2\pi j \omega_2 z_2] [1 + (q-1) \beta z_2^2]^{\frac{1}{1-q}} F_1(\omega_1, \zeta(z_2, q), q) dz_2 \quad (27)$$

By using the property that the Fourier transform of the product of two functions can be computed by the convolution in the Fourier transform domain, we can re-write expression (27) as:

$$\mathcal{F}(G_{2,q}(\mathbf{z}, \beta); \omega_1, \omega_2) = \quad (28)$$

$$= C_{2,q}(\Sigma, \beta) \sqrt{|\Sigma|} \left(\int_{-\infty}^{+\infty} \exp[-2\pi j \omega_2 z_2] [1 + (q-1) \beta z_2^2]^{\frac{1}{1-q}} dz_2 \right) \otimes \quad (29)$$

$$\left(\int_{-\infty}^{+\infty} \exp[-2\pi j \omega_2 z_2] F_1(\omega_1, a(z_2, q), q) dz_2 \right). \quad (30)$$

Unfortunately, this expression can not be analytically resolved like in the one-dimensional case. Therefore, we apply numerical methods to approximate it, as done next.

5. Numerical Computation of FT

We consider the isotropic setup ($\sigma_1 = \sigma_2 = \sigma$) and approximate expression (23) by the double summation:

$$\mathcal{F}(G_{2,q}(\mathbf{x}, \beta); \omega_1, \omega_2) \approx \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \exp[-2\pi j (\omega_1 x_m + \omega_2 y_n)] G_{2,q}(x_m, y_n; \beta) \Delta x \Delta y, \quad (31)$$

where Δx and Δy must be chosen in advance. From expressions (14)-(15) it is straightforward to show that $G_{2,q}(\mathbf{x}, \beta) \rightarrow 0$ if $\|\mathbf{x}\| \rightarrow +\infty$. Consequently, we can limit the support of $G_{2,q}$ in a rectangular region $D \subset \mathbb{R}^2$ and replace the infinite double summations by finite ones whenever $\Delta x, \Delta y > 0$. The numerical error in this process is controlled by the sizes of D and discretization parameters $\Delta x, \Delta y$. Once $G_{2,q}(\mathbf{x}, \beta)$ is a symmetric function around the origin $(0, 0)$ we can place the center of the region D in the origin $(0, 0)$. Besides, in the isotropic case we can also postulate $\Delta x = \Delta y = T$ and use a square region D . Hence, the numerical setup to compute expression (31) depends on the definition of T and the size of D . Also, in this case we have $x_m = mT$ and $y_n = nT$ in expression (31) and we can find a square D in the space domain such that $G_{2,q}(\mathbf{x}, \beta) < \delta$ if $\mathbf{x} \notin D$, for a given $\delta > 0$. By putting all these considerations together, we can re-write expression (31) as:

$$\overline{\mathcal{F}}(G_{2,q}(\mathbf{x}, \beta); \omega_1, \omega_2) \approx \sum_{m=-M}^M \sum_{n=-M}^M \exp[-2\pi j (\omega_1 x_m + \omega_2 y_n)] G_{2,q}(x_m, y_n; \beta) T^2, \quad (32)$$

where $-T^{-1} < \omega_1, \omega_2 < T^{-1}$, $x_m = mT$, $y_n = nT$, with $-M \leq m, n \leq M$. It is important to be aware to the fact that expression (32) has a period T^{-1} in both frequency directions ω_1 and ω_2 . In fact, if we compute:

$$\begin{aligned}
& \overline{\mathcal{F}}(G_{2,q}(\mathbf{x}, \beta); \omega_1 + T^{-1}, \omega_2 + T^{-1}) \\
&= \sum_{m=-M}^M \sum_{n=-M}^M \exp[-2\pi j(\omega_1 x_m + \omega_2 y_n) - 2\pi j(x_m + y_n)T^{-1}] G_{2,q}(x_m, y_n; \beta) T^2 \\
&= \sum_{m=-M}^M \sum_{n=-M}^M \exp[-2\pi j(\omega_1 x_m + \omega_2 y_n)] \exp[-2\pi j(m+n)] G_{2,q}(x_m, y_n; \beta) T^2 \\
&= \sum_{m=-M}^M \sum_{n=-M}^M \exp[-2\pi j(\omega_1 x_m + \omega_2 y_n)] G_{2,q}(x_m, y_n; \beta) T^2,
\end{aligned}$$

once m, n are integer numbers. That is why we must compute expression (32) only inside the square $-T^{-1} \leq \omega_1, \omega_2 \leq T^{-1}$ in the frequency domain.

6. Computational Experiments

Before proceeding, we shall notice that expressions (18)-(21) need some considerations before their computations. Expression (18) involves the modified Bessel functions, given by equation (95), which is not defined by $\mu \in \mathbb{Z}_-$. Also, the Gamma function, defined by expression (91) is not valid for $z \in \mathbb{Z}_-$, which imposes constraints for the Bessel functions also (see expression (96)). We must notice that the Gamma function occurs also in the normalization factor $C_{1,q}$ given in expressions (12)-(13). Therefore, if we put all the constraints together, we conclude that these expression can be computed only if:

a) Case $1 < q < 3$

$$\left\{ \frac{1}{q-1} + \frac{1}{2} \right\} \cap \mathbb{Z} = \emptyset, \text{ and } \left\{ \frac{1}{q-1} \right\} \cap \mathbb{Z}_- = \emptyset.$$

b) Case $q < 1$

$$\left\{ \frac{1}{1-q} + 1 \right\} \cap \mathbb{Z}_- = \left\{ \frac{1}{1-q} + \frac{1}{2} \right\} \cap \mathbb{Z}_- = \left\{ \frac{1}{1-q} + \frac{3}{2} \right\} \cap \mathbb{Z}_- = \emptyset.$$

The center x^* and radius Δ_ψ of a function $\psi = \psi(x)$ are defined to be:

$$x^* := \frac{1}{\|\psi\|_2^2} \int_{-\infty}^{\infty} x |\psi(x)|^2 dx, \quad (33)$$

$$\Delta_\psi \equiv \frac{1}{\|\psi\|_2} \|(x - x^*) \psi\|_2 = \frac{1}{\|\psi\|_2} \left\{ \int_{-\infty}^{\infty} (x - x^*)^2 |\psi(x)|^2 dx \right\}^{1/2}. \quad (34)$$

If $x^* < \infty$ and $\Delta_\psi < \infty$, we say that the signal ψ is *localized* about the point t^* with the space window $[x^* - \Delta_\psi, x^* + \Delta_\psi]$. The space window corresponding to the one-dimensional q -Gaussian, given by expression (11), can be computed by noticing that $x^* = 0$ due to the fact that $G_{1,q}(x, \sigma, \beta) = G_{1,q}(-x, \sigma, \beta)$. Besides, we can use a methodology that is analogous to the one applied in Appendix C to show that, for $q > 1$ we get:

$$\|G_{1,q}(x, \sigma, \beta)\|_2 \equiv \left\{ \int_{-\infty}^{\infty} |G_{1,q}(x, \sigma, \beta)|^2 dx \right\}^{1/2} = \frac{C_{1q}(\sigma, \beta)}{[(q-1) \frac{\beta}{\sigma^2}]^{1/4}} \sqrt{B\left(\frac{1}{2}, \frac{2}{q-1} - \frac{1}{2}\right)}, \quad (35)$$

with $1 < q < 3$,

$$\|x G_{1,q}(x, \sigma, \beta)\|_2 = \frac{C_{1q}(\sigma, \beta)}{[(q-1) \frac{\beta}{\sigma^2}]^{3/4}} \sqrt{B\left(\frac{3}{2}, \frac{2}{q-1} - \frac{3}{2}\right)}, \quad 1 < q < \frac{7}{3}, \quad (36)$$

$$\Delta_{G_{1,q}}^{\sigma, \beta} = \left(\frac{B\left(\frac{3}{2}, \frac{2}{q-1} - \frac{3}{2}\right)}{[(q-1) \frac{\beta}{\sigma^2}] B\left(\frac{1}{2}, \frac{2}{q-1} - \frac{1}{2}\right)} \right)^{1/2}, \quad 1 < q < \frac{7}{3}, \quad (37)$$

where $C_{1q}(\sigma, \beta)$ is given by expression (12), and B is the Beta function given by equation (92). In the section C.3 we show that for $\beta = 1/2$ the q -Gaussian generates the traditional Gaussian in the limit $q \rightarrow 1$. Therefore, to allow further comparisons, we set $\beta = 1/2$, and arbitrarily choose $\sigma = 0.1$ expression (11). The Figure 1.(a) shows the behaviour of expression (37) in the corresponding q range.

We notice that the size of the space window of $G_{1,q}(x, 0.1, 0.5)$ about $x^* = 0$ is a monotone increasing function respect to q . Consequently, we expect the behavior shown in Figure 2.(a). Also, due to the Heisenberg inequality:

$$4\pi \|x G_{1,q}(x, \sigma, \beta)\|_2 \cdot \|y \mathcal{F}(G_{1,q}(x, \sigma, \beta); y)\|_2 \geq \|G_{1,q}(x, \sigma, \beta)\|_2^2, \quad (38)$$

we get that:

$$\|y \mathcal{F}(G_{1,q}(x, \sigma, \beta); y)\|_2 \geq \Delta \mathcal{F}_{\sigma, \beta, q},$$

where:

$$\Delta \mathcal{F}_{1,q}^{\sigma, \beta} = \frac{C_{1q} [(q-1) \frac{\beta}{\sigma^2}]^{1/4}}{4\pi} \frac{B\left(\frac{1}{2}, \frac{2}{q-1} - \frac{1}{2}\right)}{\sqrt{B\left(\frac{3}{2}, \frac{2}{q-1} - \frac{3}{2}\right)}}, \quad 1 < q < \frac{7}{3}, \quad (39)$$

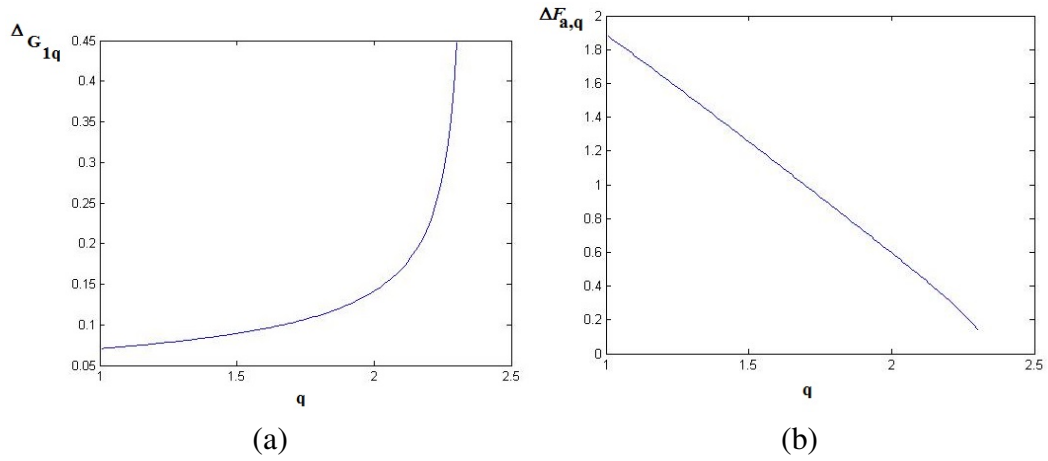


Figure 1. (a) Size of space window for q -Gaussian with parameters $\beta = 1/2, \sigma = 0.1$. (b) Behavior of expression (39) for q -Gaussian with $\beta = 1/2, \sigma = 0.1$.

and C_{1q} and B follows like in equations (35)-(37). The Figure 1.(b) indicates that the window size in the frequency domain is a monotone decreasing function respect to q . We can check this fact through Figure 2.(b) which pictures the profile of the absolute value of the Fourier transform of a q -Gaussian for three values of the entropic index q . We can notice that when increasing q the q -Gaussian becomes more localized about $y = 0$, which agrees with the decreasing behavior pictured in Figure 1.(b) for expression (39).

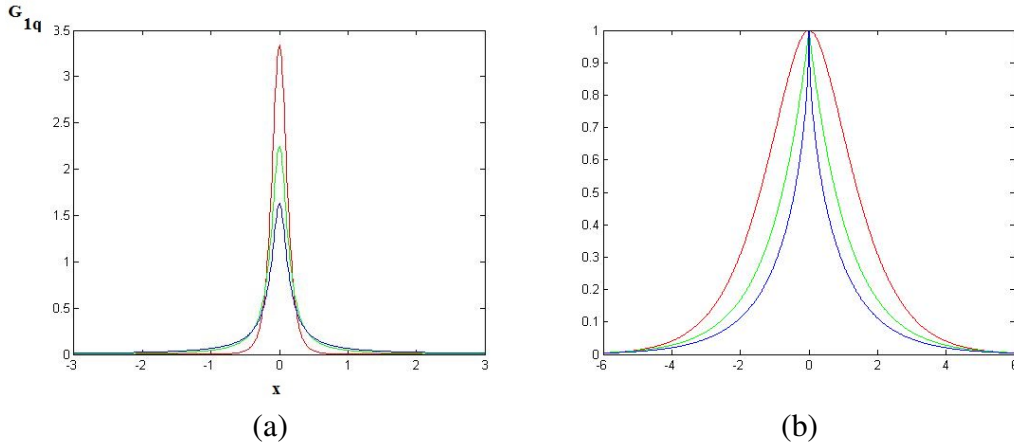


Figure 2. (a) Plot for q -Gaussian in the space domain with parameters $\beta = 1/2, \sigma = 0.1$ and $q = 1.41$ (red), $q = 2.0$ (green) and $q = 2.3$ (blue). (b) Absolute value of the Fourier transform of q -Gaussian (equation (19)) with parameters $\beta = 1/2, \sigma = 0.1$ and $q = 1.41$ (red), $q = 2.0$ (green) and $q = 2.3$ (blue).

An analogous analysis can be performed for $q < 1$. In this case, we get also $x^* = 0$ from equation (33) and we can use the same methodology applied in the Appendix C to get that:

$$\|G_{1,q}(x, \sigma, \beta)\|_2 \equiv \left\{ \int_{-\infty}^{\infty} |G_{1,q}(x, \sigma, \beta)|^2 dx \right\}^{1/2} = \frac{C_{1q}}{[(1-q) \frac{\beta}{\sigma^2}]^{1/4}} \sqrt{B\left(\frac{1}{2}, \frac{2}{1-q} + 1\right)}, \quad q < 1, \quad (40)$$

$$\|xG_{1,q}(x, \sigma, \beta)\|_2 = \frac{C_{1q}}{[(1-q) \frac{\beta}{\sigma^2}]^{3/4}} \sqrt{B\left(\frac{3}{2}, \frac{2}{1-q} + 1\right)}, \quad q < 1, \quad (41)$$

$$\Delta_{G_{1,q}}^{\sigma,\beta} = \left(\frac{B\left(\frac{3}{2}, \frac{2}{1-q} + 1\right)}{[(1-q) \frac{\beta}{\sigma^2}] B\left(\frac{1}{2}, \frac{2}{1-q} + 1\right)} \right)^{1/2}, \quad q < 1, \quad (42)$$

where $|\mathbf{x}| \leq ((1-q) \frac{\beta}{\sigma^2})^{-1/2}$, C_{1q} is given by expression (13), and B is the Beta function, computed by equation (92). By using Heisenberg inequality, given by expression (38), we can obtain:

$$\|y\mathcal{F}(G_{1,q}(x, \sigma, \beta); y)\|_2 \geq \Delta\mathcal{F}_{\sigma,\beta,q},$$

with:

$$\Delta\mathcal{F}_{1,q}^{\sigma,\beta} = \frac{C_{1q}}{4\pi} \left[(1-q) \frac{\beta}{\sigma^2} \right]^{1/4} \frac{B\left(\frac{1}{2}, \frac{2}{1-q} + 1\right)}{\sqrt{B\left(\frac{3}{2}, \frac{2}{1-q} + 1\right)}}, \quad q < 1 \quad (43)$$

The Figure 3.(a) shows the behaviour of expression (42) in the range $-2.0 < q < 0.99$ and the Figure 3.(b) pictures the behavior of expression (43) in the same range. Likewise in the $q > 1$ case, we observe that $\Delta_{G_{1,q}}^{0.5,0.1}$ and $\Delta\mathcal{F}_{1,q}^{0.5,0.1}$ are monotone increasing and monotone decreasing functions, respectively, in the considered q range.

We can check these facts through Figure 4 which pictures the q -Gaussian and its Fourier transform for $q = 0.1, 0.5, 0.99$. We can notice by Figure 4.(a) that when increasing q the q -Gaussian becomes less localized about $y = 0$, which agrees with the increasing behavior of the window size pictured in Figure 3.(a). On the other hand, the tendency for the Fourier transform localization given by expression (43), and represented in the Figure 3.(b) is confirmed by the Figure 4.(b).

The Figures 2.(b) and 4.(b) shows that: $\mathcal{F}(G_{1,q}(x, \sigma, \beta); 0) = 1$ for the considered q values. In fact, from expressions (19)-(21) we can prove this property for any $q \in \mathbb{R} - \{1\}$ and make comparisons with the (normalized) Gaussian given by expression:

$$G(x, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad (44)$$

whose Fourier transform is the function:

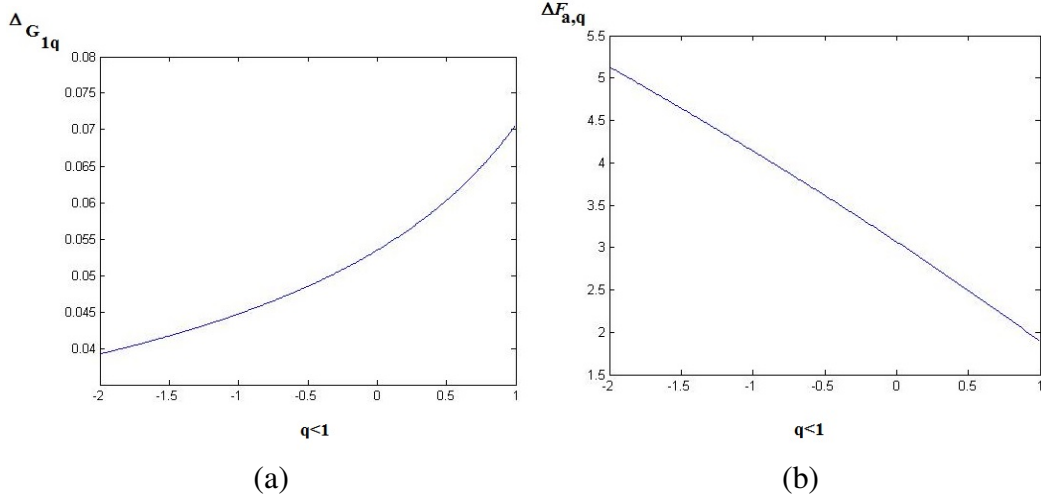


Figure 3. (a) Size of space window for q -Gaussian with parameters $\beta = 1/2, \sigma = 0.1$ for $q < 1$. (b) Behavior of expression (43) for q -Gaussian with $\beta = 1/2, \sigma = 0.1$.

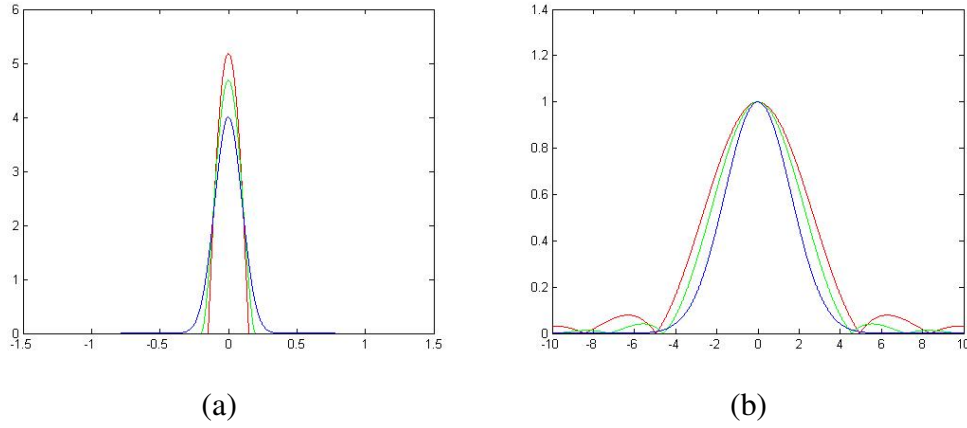


Figure 4. (a) Plot for q -Gaussian in the space domain with parameters $\beta = 1/2, \sigma = 0.1$ and $q = 0.1$ (red), $q = 0.5$ (green) and $q = 0.99$ (blue). (b) FT of q -Gaussian (denoted by $\mathcal{F}(G_{1,q}(x, a); y)$) with parameters $\beta = 1/2, \sigma = 0.1$ and $q = 0.1$ (red), $q = 0.5$ (green) and $q = 0.99$ (blue).

$$\mathcal{F}(G(x, \sigma); y) = \exp(-\sigma y^2). \quad (45)$$

So, it is clear that $\mathcal{F}(G(x, \sigma); 0) = 1$ also. Besides, Figures 2 and 4 show that, if we fix the parameters β and σ in expression (11), we can change the localization and the profile of the q -Gaussian by changing the q value. Therefore, in terms of low-pass filtering properties, the main point is how close $G_{1,q}(x, \sigma, \beta)$ (and $\mathcal{F}(G_{1,q}(x, \sigma, \beta); y)$) is from $G(x, \sigma)$ (and $\mathcal{F}(G(x, \sigma); y)$) when changing the q value in expression (45)? We must perform further developments in order to answer this question.

On the other hand, Figure 5 shows the cut-off frequency $\bar{y} = \bar{y}(q)$, such that $\text{abs}(\mathcal{F}(G_{1,q}(x, 0.1, 0.5); \bar{y})) < 0.1$.

We notice that \bar{y} is a decreasing function which is in accordance with the behaviour reported by Figures 2.(a) and 4.(a).

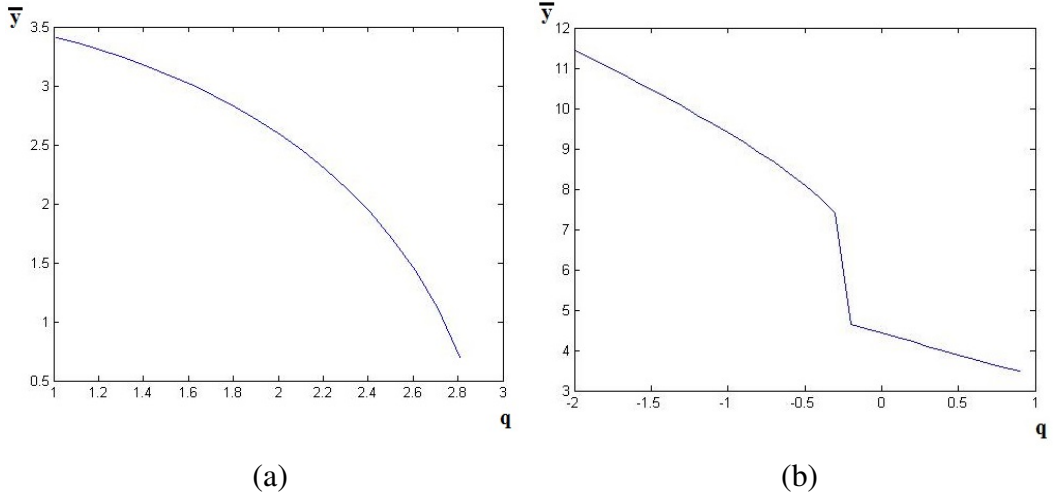


Figure 5. (a) Cut-off frequency \bar{y} such that $\text{abs}(\mathcal{F}(G_{1,q}(x, 0.1, 0.5); \bar{y})) < 0.1$ for $1 < q < 3$. (b) Cut-off frequency \bar{y} such that $\text{abs}(\mathcal{F}(G_{1,q}(x, 0.1, 0.5); \bar{y})) < 0.1$ for $-2 \leq q < 1$.

Now, we consider the FT of the q -Gaussian 2D defined by the following parameters: $\beta = 1, \sigma = \sqrt{8} \approx 2.8284, q = 0.5$. The discretization parameters used in expression (32) are: $T = 0.25, M = 2.5$. The Figure 6 pictures the obtained surface.

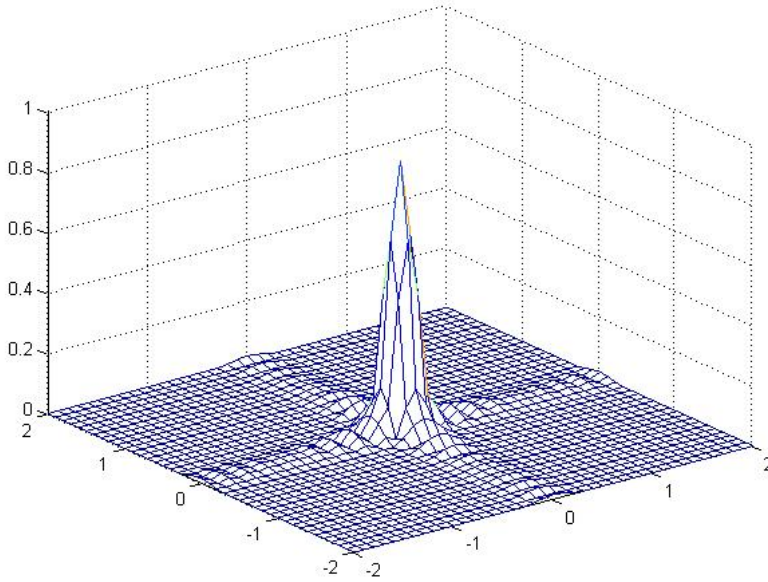


Figure 6. Plot of $\text{abs}(\mathcal{F}(G_{2,0.5}(x, y; \sqrt{8}, 1); \omega_1, \omega_2))$ for $-2 \leq \omega_1, \omega_2 < 2$.

7. Conclusions and Future Works

In this paper we collect theoretical elements to analyze the q -Gaussian kernel for feature extraction and edge detection. We review some theoretical elements behind the q -Gaussian and its Fourier transform. We analyze the q -Gaussian kernel in the space and

Fourier domains using the concepts of space window, cut-off frequency, and the Heisenberg inequality. We postulate that the comparison between the q -Gaussian and Gaussian kernels in the Fourier/space domains may allow to explain the observed smoothing capabilities of the q -Gaussian kernel for $q < 1$.

Acknowledgement

We would like to thank Dr. Ernesto Pinheiro Borges, from Federal University of Bahia, for the comments and mathematical developments, which helped us a lot to complete this work.

A. Appendix: Gaussian in \mathbb{R}^d

The d -dimensional Gaussian is defined by:

$$G(\mathbf{x}, \Sigma) = \left[(2\pi)^{d/2} |\Sigma|^{1/2} \right]^{-1} \exp \left(-\frac{1}{2} \mathbf{x}^T \Sigma^{-1} \mathbf{x} \right). \quad (46)$$

where Σ is the covariance matrix (symmetric and positive definite) and $|\Sigma|$ means its determinant.

If $d = 1$:

$$G(x, \sigma^2) = \left[\sqrt{2\pi\sigma^2} \right]^{-1} \exp \left(-\frac{x^2}{2\sigma^2} \right) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left(-\frac{x^2}{2\sigma^2} \right)$$

Analogously, if $d = 2$ then $\mathbf{x} = (x, y)$ and expression (46) renders:

$$G(x, y, \Sigma) = \left[(2\pi) |\Sigma|^{1/2} \right]^{-1} \exp \left(-\frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \Sigma^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right).$$

If the covariance matrix is diagonal:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} \implies \Sigma^{-1} = \begin{pmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{pmatrix}.$$

So:

$$G(x, y, \Sigma) = \frac{1}{2\pi\sigma_1\sigma_2} \exp \left(-\frac{1}{2} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \sigma_1^{-2} & 0 \\ 0 & \sigma_2^{-2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \right),$$

and:

$$G(x, y, \Sigma) = \frac{1}{2\pi\sigma_1\sigma_2} \exp \left(-\frac{1}{2} \left(\frac{x^2}{\sigma_1^2} + \frac{y^2}{\sigma_2^2} \right) \right).$$

B. Appendix: q -Gaussian in \mathbb{R}^d

The d -dimensional q -Gaussian is defined by:

$$G_{d,q}(\mathbf{x}, \Sigma, \beta) = C_{d,q}(\Sigma, \beta) \exp_q(-\beta \mathbf{x}^T \Sigma^{-1} \mathbf{x}), \quad (47)$$

where $C_{d,q}(\Sigma, \beta)$ is a normalization factor.

Once the covariance matrix Σ is symmetric and positive definite, there exists an orthogonal matrix U and a diagonal matrix D such that:

$$\Sigma = U^T D U.$$

Let

$$\mathbf{y} = U \mathbf{x}, \quad (48)$$

Then $\mathbf{x} = U^T \mathbf{y}$ and, once $\text{abs}|U| = 1$:

$$C_{d,q}(\Sigma, \beta) = \left(\int_{\mathbb{R}^d} \exp_q(-\beta \mathbf{y}^T U \Sigma^{-1} U^T \mathbf{y}) d\mathbf{y} \right)^{-1}.$$

But:

$$\Sigma^{-1} = (U^T D U)^{-1} = U^T D^{-1} U \implies D^{-1} = U \Sigma^{-1} U^T. \quad (49)$$

Therefore:

$$C_{d,q}(\Sigma, \beta) = \left(\int_{\mathbb{R}^d} \exp_q(-\beta \mathbf{y}^T D^{-1} \mathbf{y}) d\mathbf{y} \right)^{-1}.$$

If:

$$D^{-1} = \begin{pmatrix} \sigma_1^{-2} & & & \\ & \sigma_2^{-2} & & \\ & & \dots & \\ & & & \sigma_d^{-2} \end{pmatrix},$$

then: $D^{-1} = \sqrt{D^{-1}} \sqrt{D^{-1}}$, where:

$$\sqrt{D^{-1}} = \begin{pmatrix} \sigma_1^{-1} & & & \\ & \sigma_2^{-1} & & \\ & & \dots & \\ & & & \sigma_d^{-1} \end{pmatrix}.$$

If

$$\mathbf{z} = \sqrt{D^{-1}}\mathbf{y}, \quad (50)$$

then

$$d\mathbf{z} = \det\left(\sqrt{D^{-1}}\right) d\mathbf{y},$$

and;

$$\det\left(\sqrt{D^{-1}}\right) = \det\left(D^{-1/2}\right) = \left(\frac{1}{\sqrt{\det(D)}}\right) = \frac{1}{\sqrt{\det(U\Sigma U^T)}} = \frac{1}{\sqrt{\det(\Sigma)}} \equiv \frac{1}{\sqrt{|\Sigma|}}.$$

Therefore:

$$d\mathbf{y} = \sqrt{|\Sigma|}d\mathbf{z}.$$

Consequently:

$$C_{d,q}(\Sigma, \beta) = \left(\int_{\mathbb{R}^d} \exp_q(-\beta \mathbf{y}^T D^{-1} \mathbf{y}) d\mathbf{y}\right)^{-1} = \left(\int_{\mathbb{R}^d} \exp_q(-\beta |\mathbf{z}|^2) \sqrt{|\Sigma|} d\mathbf{z}\right)^{-1} \quad (51)$$

$$= \frac{1}{\sqrt{|\Sigma|} \int_{\mathbb{R}^d} \exp_q(-\beta |\mathbf{z}|^2) d\mathbf{z}}. \quad (52)$$

C. Appendix: One-Dimensional q -Gaussian

If $d = 1$ in expression (52) we get:

$$C_{1,q}(\sigma, \beta) = \left(\int_{\mathbb{R}} \exp_q(-\beta y \sigma^{-2} y) dy\right)^{-1} = \left(\int_{\mathbb{R}} \exp_q\left(-\frac{\beta}{\sigma^2} y^2\right) dy\right)^{-1} = \frac{1}{\int_{\mathbb{R}} \exp_q\left(-\frac{\beta}{\sigma^2} y^2\right) dy}, \quad (53)$$

where:

$$\begin{aligned} \int_{\mathbb{R}} \exp_q\left(-\frac{\beta}{\sigma^2} y^2\right) dy &= \int_{\mathbb{R}} \left[1 + (1 - q) \left(-\frac{\beta}{\sigma^2} y^2\right)\right]^{\frac{1}{1-q}} dy \\ &= \int_{\mathbb{R}} \left[1 + (q - 1) \left(\frac{\beta}{\sigma^2} y^2\right)\right]^{\frac{1}{1-q}} dy. \end{aligned} \quad (54)$$

C.1. Case $d = 1$ and $q > 1$

If

$$\tilde{y} = [(q-1)a]^{1/2} y,$$

then:

$$\int_{\mathbb{R}} \exp_q \left(-\frac{\beta}{\sigma^2} y^2 \right) dy = \int_{\mathbb{R}} [1 + (q-1)(ay^2)]^{\frac{1}{1-q}} dy = \frac{2}{[(q-1)a]^{1/2}} \int_0^{+\infty} [1 + \tilde{y}^2]^{\frac{1}{1-q}} d\tilde{y}. \quad (55)$$

From equations 3.251-2 and 8.384 of reference [Gradshteyn and Ryzhik 1981], we get:

$$\int_0^{+\infty} x^{\mu-1} [1+x^2]^{\nu-1} dx = \frac{1}{2} B \left(\frac{\mu}{2}, 1-\nu-\frac{\mu}{2} \right), \quad \text{if } \operatorname{Re}(\mu) > 0, \quad \operatorname{Re} \left(\nu + \frac{1}{2}\mu \right) < 1, \quad (56)$$

where B is the Beta function (see equation (92)). Therefore, we can cast expression (55) in the above form by setting:

$$\mu = 1, \quad \nu - 1 = \frac{1}{1-q},$$

which implies that:

$$\nu = \frac{1}{1-q} + 1 = \frac{2-q}{1-q}.$$

By inserting these values in the conditions for expression (56) we get:

$$\frac{1}{1-q} + 1 + \frac{1}{2} < 1,$$

and, consequently:

$$1 < q < 3.$$

So, we can insert the above results in expression (56) to obtain:

$$\int_0^{+\infty} [1 + \tilde{y}^2]^{\frac{1}{1-q}} d\tilde{y} = \int_0^{+\infty} [1 + x^2]^{\frac{1}{1-q}} dx = \frac{1}{2} B \left(\frac{1}{2}, \frac{1}{q-1} - \frac{1}{2} \right),$$

where, by using the Beta function definition given by expression (92), it can be written:

$$B \left(\frac{1}{2}, \frac{1}{q-1} - \frac{1}{2} \right) = \frac{\Gamma \left(\frac{1}{2} \right) \Gamma \left(\frac{1}{q-1} - \frac{1}{2} \right)}{\Gamma \left(\frac{1}{q-1} \right)}.$$

Through this result and the fact $\Sigma = \sigma^2$ for $d = 1$, we can compute expression (53) as:

$$\begin{aligned} C_{1,q}(\sigma, \beta) &= \frac{1}{\int_{\mathbb{R}} \exp_q \left(-\frac{\beta}{\sigma^2} y^2 \right) dy} \\ &= \frac{1}{\frac{2}{[(q-1)a]^{1/2}} \frac{1}{2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{q-1}-\frac{1}{2})}{\Gamma(\frac{1}{q-1})}} = \frac{\Gamma\left(\frac{1}{q-1}\right) [(q-1)a]^{1/2}}{\sqrt{\pi}\Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}, \end{aligned} \quad (57)$$

once $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (see [Filho 1986], page 37).

C.2. Case $d = 1$ and $q < 1$

In this case, expression (54) is well defined only if:

$$1 + (q-1)ay \geq 0,$$

which implies:

$$|y| \leq \frac{1}{\sqrt{(1-q)a}} = ((1-q)a)^{-1/2}.$$

Hence, equation (54) becomes:

$$\int_{\mathbb{R}} [1 + (q-1)(ay^2)]^{\frac{1}{1-q}} dy = \int_{-((1-q)a)^{-1/2}}^{+((1-q)a)^{-1/2}} [1 + (q-1)(ay^2)]^{\frac{1}{1-q}} dy.$$

By using the variable change:

$$\tilde{y} = ((1-q)a)^{1/2} y,$$

we get:

$$\int_{-1}^{+1} [1 - \tilde{y}^2]^{\frac{1}{1-q}} \frac{d\tilde{y}}{((1-q)a)^{1/2}}. \quad (58)$$

From equation 3.251-1 of reference [Gradshteyn and Ryzhik 1981], we see:

$$\int_0^1 x^{\mu-1} (1-x^\lambda)^{\nu-1} dx = \frac{1}{\lambda} B\left(\frac{\mu}{\lambda}, \nu\right), \quad \text{if } \operatorname{Re}(\mu) > 0, \operatorname{Re}(\nu) > 0, \lambda > 0, \quad (59)$$

where B is the Beta function given by expression (92).

Therefore, by setting:

$$\mu = 1, \lambda = 2, \nu = \frac{1}{1-q} + 1,$$

in expression (59) we obtain:

$$\int_0^1 (1-x^2)^{\frac{1}{1-q}} dx = \frac{1}{2} B\left(\frac{1}{2}, \frac{1}{1-q} + 1\right).$$

Hence, we can compute equation (58) as:

$$\begin{aligned} & \frac{1}{((1-q)a)^{1/2}} \int_{-1}^{+1} [1-\tilde{y}^2]^{\frac{1}{1-q}} d\tilde{y} \\ & \frac{2}{((1-q)a)^{1/2}} \int_0^1 (1-x^2)^{\frac{1}{1-q}} dx = \frac{1}{((1-q)a)^{1/2}} B\left(\frac{1}{2}, \frac{1}{1-q} + 1\right). \end{aligned}$$

By inserting this result and equation (92) in expression (53) and we obtain:

$$\begin{aligned} C_{1,q}(\beta) &= \frac{1}{\int_{\mathbb{R}} \exp_q\left(-\frac{\beta}{\sigma^2} y^2\right) dy} \\ &= \frac{1}{\frac{1}{((1-q)a)^{1/2}} B\left(\frac{1}{2}, \frac{1}{1-q} + 1\right)} \\ &= \frac{1}{\frac{1}{((1-q)a)^{1/2}} B\left(\frac{1}{2}, \frac{1}{1-q} + 1\right)} \\ &= \frac{1}{\frac{1}{((1-q)a)^{1/2}} \frac{\Gamma(\frac{1}{2}) \Gamma(\frac{1}{1-q} + 1)}{\Gamma(\frac{1}{1-q} + \frac{3}{2})}} \\ &= \frac{\Gamma\left(\frac{1}{1-q} + \frac{3}{2}\right) ((1-q)a)^{1/2}}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{1-q} + 1\right)}. \end{aligned} \tag{60}$$

for $q < 1$.

C.3. One-Dimensional q -Gaussian Expression

By inserting equations (57) and (60) in expression (7) with $d = 1$ we get:

$$G_{1,q}(x, \sigma, \beta) = C_{1,q}(\sigma, \beta) \exp_q\left(-\frac{\beta}{\sigma^2} x^2\right), \tag{61}$$

where:

$$C_{1,q}(\sigma, \beta) = \frac{\Gamma\left(\frac{1}{q-1}\right) [(q-1) \frac{\beta}{\sigma^2}]^{1/2}}{\sqrt{\pi} \Gamma\left(\frac{1}{q-1} - \frac{1}{2}\right)}, \quad 3 > q > 1,$$

$$C_{1,q}(\sigma, \beta) = \frac{\Gamma\left(\frac{1}{1-q} + \frac{3}{2}\right) ((1-q) \frac{\beta}{\sigma^2})^{1/2}}{\sqrt{\pi} \Gamma\left(\frac{1}{1-q} + 1\right)}, \quad q < 1, \quad |x| \leq \left((1-q) \frac{\beta}{\sigma^2}\right)^{-1/2}.$$

Consequently, using equation A7 of reference [Borges 2012], which states that:

$$\lim_{q \rightarrow 1^+} \frac{\Gamma\left(\frac{1}{q-1} - \alpha\right)}{(q-1)^\alpha \Gamma\left(\frac{1}{q-1}\right)} = 1, \quad 1 < q < 1 + \frac{1}{\alpha},$$

$$\lim_{q \rightarrow 1^-} \frac{\Gamma\left(\frac{1}{1-q} + 1\right)}{(1-q)^\alpha \Gamma\left(\frac{1}{1-q} + \alpha + 1\right)} = 1, \quad q < 1,$$

we can show that:

$$\lim_{q \rightarrow 1} G_{1,q}(\mathbf{x}, \sigma, \beta) = \frac{\sqrt{\beta}}{\sigma \sqrt{\pi}} \exp\left(-\frac{\beta}{\sigma^2} x^2\right).$$

If $\beta = 0.5$, we get:

$$\lim_{q \rightarrow 1} G_{1,q}(\mathbf{x}, \sigma, 0.5) = \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad (62)$$

which is the traditional Gaussian.

D. Appendix: Two-Dimensional q -Gaussian

If $d = 2$ then we can write $\mathbf{z} = (z_1, z_2)$ and, using polar coordinates:

$$z_1 = r \cos(\theta), \quad (63)$$

$$z_2 = r \sin(\theta), \quad (64)$$

$$\det(J) = r,$$

we can write the integral in expression (52) as:

$$\int_{\mathbb{R}^2} \exp_q(-\beta |\mathbf{z}|^2) d\mathbf{z} = \int_0^{2\pi} \int_0^{+\infty} r \exp_q(-\beta r^2) dr d\theta.$$

In the above development we are supposing $\beta > 0$.

D.1. Case $q > 1$

Then:

$$\begin{aligned}
 \int_0^{2\pi} \int_0^{+\infty} r \exp_q(-\beta r^2) dr d\theta &= 2\pi \int_0^{+\infty} r \exp_q(-\beta r^2) dr \\
 &= 2\pi \int_0^{+\infty} r [1 + (1-q)(-\beta r^2)]^{\frac{1}{1-q}} dr \\
 &= 2\pi \int_0^{+\infty} r [1 - (1-q)\beta r^2]^{\frac{1}{1-q}} dr.
 \end{aligned} \tag{65}$$

Variable change:

$$u = 1 - (1-q)\beta r^2, \tag{66}$$

$$du = -2(1-q)\beta r dr = 2(q-1)\beta r dr. \tag{67}$$

Besides, once $q > 1$ and $\beta > 0$, we have

$$r \rightarrow +\infty \implies u \rightarrow +\infty, \quad \text{and} \quad u(0) = 1.$$

Therefore, using the variable change defined by expressions (66)-(67) we obtain:

$$\begin{aligned}
 2\pi \int_0^{+\infty} r [1 - (1-q)\beta r^2]^{\frac{1}{1-q}} dr &= \frac{2\pi}{2(q-1)\beta} \int_0^{+\infty} [1 - (1-q)\beta r^2]^{\frac{1}{1-q}} (2(q-1)\beta r dr) \\
 &= \frac{2\pi}{2(q-1)\beta} \int_1^{+\infty} u^{\frac{1}{1-q}} du = \frac{2\pi}{2(q-1)\beta} \left[\frac{u^{\frac{1}{1-q}+1}}{\frac{1}{1-q}+1} \right]_1^{+\infty} = \frac{2\pi}{2(q-1)\beta} \left[\frac{u^{\frac{2-q}{1-q}}}{\frac{2-q}{1-q}} \right]_1^{+\infty}.
 \end{aligned}$$

The above integral converges only if:

$$\frac{1}{1-q} + 1 = \frac{2-q}{1-q} < 0 \implies q < 2.$$

Once we are considering $q > 1$, we get that, if $1 < q < 2$ then:

$$2\pi \int_0^{+\infty} r [1 - (1-q)\beta r^2]^{\frac{1}{1-q}} dr = \frac{2\pi}{2(q-1)\beta} \left(-\frac{1}{\frac{2-q}{1-q}} \right) = \frac{2\pi}{2(q-1)\beta} \left(\frac{q-1}{2-q} \right) = \frac{\pi}{\beta(2-q)}. \tag{68}$$

D.2. Case $q < 1$

In this case, in order to get a real value in the integral given by expression (65) we need:

$$1 - (1 - q) \beta r^2 > 0 \implies 0 < r < \frac{1}{\sqrt{\beta(1 - q)}}. \quad (69)$$

Then, with the constraints $q < 1$ and $0 < r < (\beta(1 - q))^{-1/2}$, we can use the same variable change as before:

$$u = u(r) = 1 + (q - 1) \beta r^2 > 0.$$

By inserting it in the integral (65) we obtain:

$$\begin{aligned} 2\pi \int_0^{+\frac{1}{\sqrt{1-q}}} r [1 - (1 - q) r^2]^{\frac{1}{1-q}} dr &= \left(\frac{2\pi}{2(q - 1)\beta} \right) \left[\frac{u^{\frac{2-q}{1-q}}}{\frac{2-q}{1-q}} \right]_1^0 \\ &= \left(\frac{2\pi}{2(q - 1)\beta} \right) \left[0 - \frac{(1 - q)}{(2 - q)} \right] = \frac{\pi}{\beta(2 - q)}. \end{aligned}$$

D.3. q -Gaussian 2D: Putting All Together

Therefore, for $d = 2$, expression (8) gives:

$$C_{2,q}(\Sigma, \beta) = \left(\int_{\mathbb{R}^2} \exp_q(-\beta \mathbf{y}^T D^{-1} \mathbf{y}) d\mathbf{y} \right)^{-1} = \left(\int_{\mathbb{R}^2} \exp_q(-\beta |\mathbf{z}|^2) \sqrt{|\Sigma|} d\mathbf{z} \right)^{-1} \quad (70)$$

$$= \frac{1}{\sqrt{|\Sigma|} \int_{\mathbb{R}^2} \exp_q(-\beta |\mathbf{z}|^2) d\mathbf{z}} \quad (71)$$

Then, for $\beta > 0$, expression (68) renders:

$$\int_{\mathbb{R}^2} \exp_q(-\beta |\mathbf{z}|^2) d\mathbf{z} = \frac{\pi}{\beta(2 - q)}, \quad \text{if } 1 < q < 2, \quad (72)$$

If $q < 1$, we have the restriction:

$$\Omega = \left\{ r \in \mathbb{R}; 0 < r < \frac{1}{\sqrt{\beta(1 - q)}} \right\}, \quad (73)$$

due to expression (69). Therefore, according to section D.2, the integral in equation (71) becomes:

$$\int_{\mathbb{R}^2} \exp_q(-\beta|\mathbf{z}|^2) d\mathbf{z} = \int_0^{2\pi} \int_{\Omega} r \exp_q(-\beta r^2) dr d\theta = \frac{\pi}{\beta(2-q)}.$$

By assembling all the above results we obtain that, for $d = 2$, expression (47) becomes:

$$G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta(2-q)}{\pi\sqrt{|\Sigma|}} [1 + (1-q)(-\beta\mathbf{x}^T\Sigma^{-1}\mathbf{x})]^{\frac{1}{1-q}}, \quad 1 < q < 2, \quad (74)$$

$$G_{2,q}(\mathbf{x}, \Sigma, \beta) = \frac{\beta(2-q)}{\pi\sqrt{|\Sigma|}} [1 + (1-q)(-\beta\mathbf{x}^T\Sigma^{-1}\mathbf{x})]^{\frac{1}{1-q}}, \quad q < 1, \quad (75)$$

subject to the constraint:

$$0 < (\mathbf{x}^T\Sigma^{-1}\mathbf{x})^{1/2} < \frac{1}{\sqrt{\beta(1-q)}}.$$

Consequently:

$$\lim_{q \rightarrow 1} G_{2,q}(\mathbf{x}, \Sigma, \beta) = \left[\lim_{q \rightarrow 1} C_{2,q}(\mathbf{x}, \beta) \right] \left[\lim_{q \rightarrow 1} \exp_q(-\beta\mathbf{x}^T\Sigma^{-1}\mathbf{x}) \right] = \frac{\beta}{\pi\sqrt{|\Sigma|}} \exp(-\beta\mathbf{x}^T\Sigma^{-1}\mathbf{x}).$$

So, by setting $\beta = 1/2$, we get:

$$\lim_{q \rightarrow 1} G_{2,q}\left(\mathbf{x}, \Sigma, \frac{1}{2}\right) = \left[\lim_{q \rightarrow 1} C_{2,q}\left(\mathbf{x}, \frac{1}{2}\right) \right] \left[\lim_{q \rightarrow 1} \exp_q\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right) \right] = \frac{1}{2\pi\sqrt{|\Sigma|}} \exp\left(-\frac{1}{2}\mathbf{x}^T\Sigma^{-1}\mathbf{x}\right),$$

which is the traditional two dimensional Gaussian.

E. Appendix: Fourier Transform of q -Exponential

Let:

$$\begin{aligned} \mathcal{F}(\exp_q(-ax^2); y) &= \int_{-\infty}^{+\infty} \exp(-2j\pi xy) [1 - (1-q)ax^2]^{\frac{1}{1-q}} dx \\ &= \int_{-\infty}^{+\infty} \exp(-2j\pi xy) [1 + (q-1)ax^2]^{\frac{1}{1-q}} dx. \end{aligned} \quad (76)$$

Variable change:

$$\tilde{x} = [(q-1)a]^{1/2} x; \quad \tilde{y} = [(q-1)a]^{-1/2} y. \quad (77)$$

Then:

$$\mathcal{F}(\exp_q(-ax^2); y) = [(q-1)a]^{-1/2} \int_{-\infty}^{+\infty} \exp(-2j\pi\tilde{x}\tilde{y}) [1 + \tilde{x}^2]^{\frac{1}{1-q}} d\tilde{x}. \quad (78)$$

We are going to use the fact that:

$$\begin{aligned} & \int_{-\infty}^{+\infty} (\beta + jx)^{-2\mu} (\gamma - jx)^{-2\nu} e^{-jpx} dx \\ &= -2\pi (\beta + \gamma)^{-\mu-\nu} \frac{p^{\mu+\nu-1}}{\Gamma(2\nu)} \exp\left(\frac{\gamma - \beta}{2}p\right) W_{\mu-\nu, \frac{1}{2}-\nu-\mu}(\beta p + \gamma p), \quad p > 0, \\ &= 2\pi (\beta + \gamma)^{-\mu-\nu} \frac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)} \exp\left(\frac{\beta - \gamma}{2}p\right) W_{\mu-\nu, \frac{1}{2}-\nu-\mu}(-\beta p - \gamma p), \quad p < 0, \end{aligned}$$

if $Re(\beta) > 0$, $Re(\gamma) > 0$ and $Re(\mu + \nu) > 1/2$, where W denotes the Whittaker functions (see expressions 9-220 of reference [Gradshteyn and Ryzhik 1981]).

Therefore, if $\beta = \gamma = 1$ and $\mu = \nu$ we have:

$$(\beta + jx)^{-2\mu} (\gamma - jx)^{-2\nu} = (1 + jx)^{-2\mu} (1 - jx)^{-2\mu} = (1 + x^2)^{-2\mu},$$

and:

$$\begin{aligned} & \int_{-\infty}^{+\infty} (1 + x^2)^{-2\mu} e^{-jpx} dx \\ &= -2\pi (2)^{-2\mu} \frac{p^{2\mu-1}}{\Gamma(2\mu)} W_{0, \frac{1}{2}-2\mu}(2p), \quad p > 0, \end{aligned} \quad (79)$$

$$= 2\pi (2)^{-2\mu} \frac{(-p)^{2\mu-1}}{\Gamma(2\mu)} W_{0, \frac{1}{2}-2\mu}(-2p), \quad p < 0, \quad (80)$$

where $W_{0, \frac{1}{2}-2\mu}$ is defined by expression (93).

So, we can put expressions (79)-(80) together to obtain:

$$\int_{-\infty}^{+\infty} (1 + x^2)^{-2\mu} e^{-jpx} dx = -\text{sign}(p) 2\pi (2^{-2\mu}) \frac{|p|^{2\mu-1}}{\Gamma(2\mu)} W_{0, \frac{1}{2}-2\mu}(2|p|) \quad (81)$$

$$= -\text{sign}(p) 2\pi \frac{1}{2} \frac{1}{2^{2\mu-1}} \frac{|p|^{2\mu-1}}{\Gamma(2\mu)} W_{0, \frac{1}{2}-2\mu}(2|p|) \quad (82)$$

$$= -\text{sign}(p) \pi \frac{1}{\Gamma(2\mu)} \left(\frac{|p|}{2}\right)^{2\mu-1} W_{0, \frac{1}{2}-2\mu}(2|p|), \quad (83)$$

if $\text{Re}(2\mu) > 1/2$.

Now, let us return to expression (78). By setting: $p = 2\pi\tilde{y}$ and $2\mu = (q-1)^{-1}$, with the constraint:

$$\text{Re}(2\mu) > 1/2 \implies (q-1)^{-1} > \frac{1}{2}, \quad (84)$$

we get the following cases.

E.1. Case $q > 1$

Due to the restriction (84), we have $1 < q < 3$. By inserting equation (83) into expression (78) we obtain:

$$\begin{aligned} \mathcal{F}(\exp_q(-ax^2); y) &= [(q-1)a]^{-1/2} \int_{-\infty}^{+\infty} \exp(-2j\pi\tilde{x}\tilde{y}) [1 + \tilde{x}^2]^{\frac{1}{1-q}} d\tilde{x} \\ &= [(q-1)a]^{-1/2} \left(-\text{sign}(2\pi\tilde{y}) \pi \frac{1}{\Gamma\left(\frac{1}{q-1}\right)} \left(\frac{|2\pi\tilde{y}|}{2}\right)^{2\mu-1} W_{0, \frac{1}{2}-\frac{1}{q-1}}(2|2\pi\tilde{y}|) \right) \\ &= [(q-1)a]^{-1/2} \left(-\text{sign}\left(2\pi[(q-1)a]^{-1/2}y\right) 2\pi \left(2^{\frac{1}{1-q}}\right) \frac{|2\pi[(q-1)a]^{-1/2}y|^{\frac{1}{q-1}-1}}{\Gamma\left(\frac{1}{q-1}\right)} \right) \times \\ &\quad W_{0, \frac{1}{2}-\frac{1}{q-1}}\left(2|2\pi[(q-1)a]^{-1/2}y|\right), \end{aligned} \quad (85)$$

where the function $W_{0, \frac{1}{2}-\frac{1}{q-1}}$ is defined through equation (93). Using the fact that:

$$W_{0, \mu}(z) = W_{0, -\mu}(z),$$

we can write:

$$\begin{aligned} &|\mathcal{F}(\exp_q(-ax^2); y)| \\ &= \text{abs} \left[[(q-1)a]^{-1/2} \left(2\pi \left(2^{\frac{1}{1-q}}\right) \frac{|2\pi[(q-1)a]^{-1/2}y|^{\frac{1}{q-1}-1}}{\Gamma\left(\frac{1}{q-1}\right)} \right) \right] \times \end{aligned}$$

$$abs \left(\frac{2|2\pi [(q-1)a]^{-1/2} y|}{\pi} \right)^{1/2} \times$$

$$abs \left[\frac{i\pi}{2} \exp \left(\frac{i\pi}{2} \left(-\frac{1}{2} - \frac{1}{1-q} \right) \right) H_{-\frac{1}{2}-\frac{1}{1-q}}^{1-\frac{1}{2}-\frac{1}{1-q}} \left(i \frac{2|2\pi [(q-1)a]^{-1/2} y|}{2} \right) \right].$$

E.2. Case $q < 1$

In this case, expression (76) is well defined only if:

$$1 + (q-1)ax^2 \geq 0,$$

which implies:

$$|x| \leq \frac{1}{\sqrt{(1-q)a}} = ((1-q)a)^{-1/2}.$$

Hence, equation (76) becomes:

$$\int_{-((1-q)a)^{-1/2}}^{+((1-q)a)^{-1/2}} \exp(-2j\pi xy) [1 + (q-1)ax^2]^{\frac{1}{1-q}} dx. \quad (86)$$

By using the variable change:

$$\tilde{x} = ((1-q)a)^{1/2} x,$$

$$\tilde{y} = ((1-q)a)^{-1/2} y,$$

we can rewrite equation (86) as:

$$\int_{-1}^{+1} \exp(-2j\pi\tilde{x}\tilde{y}) [1 - \tilde{x}^2]^{\frac{1}{1-q}} \frac{d\tilde{x}}{((1-q)a)^{1/2}}. \quad (87)$$

However, it is known that (equation 3.387-2 of [Gradshteyn and Ryzhik 1981]):

$$\int_{-1}^{+1} (1-x^2)^{\nu-1} \exp(j\mu x) dx = \sqrt{\pi} \left(\frac{2}{\mu} \right)^{\nu-\frac{1}{2}} \Gamma(\nu) \mathbb{J}_{\nu-\frac{1}{2}}(\mu), \quad (88)$$

if $Re(\nu) > 0$.

Therefore, if we set:

$$\nu = \frac{1}{1-q} + 1,$$

$$\mu = -2\pi\tilde{y},$$

$$x = \tilde{x},$$

in expression (88), we obtain $Re(\nu) > 0$, and:

$$\begin{aligned} & \int_{-1}^{+1} (1 - \tilde{x}^2)^{\frac{1}{1-q}} \exp(-2j\pi\tilde{x}\tilde{y}) d\tilde{x} \\ &= \sqrt{\pi} \left(-\frac{1}{\pi\tilde{y}} \right)^{\frac{1}{1-q} + \frac{1}{2}} \Gamma \left(\frac{1}{1-q} + 1 \right) \mathbb{J}_{\frac{1}{1-q} + \frac{1}{2}}(-2\pi\tilde{y}), \text{ if } y \neq 0, \end{aligned}$$

where \mathbb{J} denotes the Bessel functions (see section F).

Consequently, we finally have:

$$\begin{aligned} \mathcal{F}(\exp_q(-ax^2); y) &= \int_{-((1-q)a)^{-1/2}}^{+((1-q)a)^{-1/2}} \exp(-2j\pi xy) [1 + (q-1)ax^2]^{\frac{1}{1-q}} dx \\ &= \frac{1}{((1-q)a)^{1/2}} \int_{-1}^{+1} \exp(-2j\pi\tilde{x}\tilde{y}) [1 - \tilde{x}^2]^{\frac{1}{1-q}} d\tilde{x} \\ &= \frac{1}{((1-q)a)^{1/2}} \sqrt{\pi} \left(-\frac{1}{\pi\tilde{y}} \right)^{\frac{1}{1-q} + \frac{1}{2}} \Gamma \left(\frac{1}{1-q} + 1 \right) \mathbb{J}_{\frac{1}{1-q} + \frac{1}{2}}(-2\pi\tilde{y}) \\ &= \frac{\sqrt{\pi}}{((1-q)a)^{1/2}} \left(-\frac{1}{\pi \left[((1-q)a)^{-1/2} y \right]} \right)^{\frac{1}{1-q} + \frac{1}{2}} \Gamma \left(\frac{1}{1-q} + 1 \right) \mathbb{J}_{\frac{1}{1-q} + \frac{1}{2}} \left(-2\pi \left[((1-q)a)^{-1/2} y \right] \right) \\ &= \frac{\sqrt{\pi}}{((1-q)a)^{1/2}} \Gamma \left(\frac{1}{1-q} + 1 \right) \left(-\frac{((1-q)a)^{1/2}}{\pi y} \right)^{\frac{1}{1-q} + \frac{1}{2}} \mathbb{J}_{\frac{1}{1-q} + \frac{1}{2}} \left(-\frac{2\pi y}{((1-q)a)^{1/2}} \right), \end{aligned} \tag{89}$$

if $q < 1$ and $y \neq 0$.

If $y = 0$ in expression (86) we get:

$$\int_{-((1-q)a)^{-1/2}}^{+((1-q)a)^{-1/2}} [1 + (q-1)ax^2]^{\frac{1}{1-q}} dx$$

$$= \int_{-1}^{+1} [1 - \tilde{x}^2]^{\frac{1}{1-q}} \frac{d\tilde{x}}{((1-q)a)^{1/2}}. \quad (90)$$

However, for expression 3.214, reference [Gradshteyn and Ryzhik 1981], we have:

$$\int_0^1 [(1+x)^{\mu-1} (1-x)^{\nu-1} + (1+x)^{\nu-1} (1-x)^{\mu-1}] dx = 2^{\mu+\nu-1} B(\mu, \nu), \quad \text{Re}(\mu) > 0, \text{Re}(\nu) > 0.$$

Therefore, if :

$$\mu = \nu = \frac{1}{(1-q)} + 1,$$

we satisfy $\text{Re}(\mu) > 0, \text{Re}(\nu) > 0$ and:

$$2 \int_0^1 [(1+x)^{\frac{1}{(1-q)}} (1-x)^{\frac{1}{(1-q)}}] dx = 2 \int_0^1 (1-x^2)^{\frac{1}{(1-q)}} dx = 2^{\frac{2}{1-q}+1} B\left(\frac{1}{(1-q)} + 1, \frac{1}{(1-q)} + 1\right).$$

So, expression (90) can be computed by:

$$\begin{aligned} & \mathcal{F}(\exp_q(-ax^2); 0) \\ &= \frac{1}{((1-q)a)^{1/2}} \int_{-1}^{+1} [1 - \tilde{x}^2]^{\frac{1}{1-q}} d\tilde{x} = \frac{2^{\frac{2}{1-q}+1}}{((1-q)a)^{1/2}} B\left(\frac{1}{(1-q)} + 1, \frac{1}{(1-q)} + 1\right) \\ &= \frac{2^{\frac{2}{1-q}+1}}{((1-q)a)^{1/2}} \frac{\Gamma\left(\frac{1}{(1-q)} + 1\right) \Gamma\left(\frac{1}{(1-q)} + 1\right)}{\Gamma\left(\frac{2}{1-q} + 2\right)}. \end{aligned}$$

F. Appendix: Special Functions

- Gamma Function: Defined by Euler in the form of the product:

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{+\infty} \left\{ \left(1 + \frac{1}{n}\right)^z \left(1 + \frac{z}{n}\right)^{-1} \right\}, \quad (91)$$

which is not valid for $z \in \mathbb{Z}_-$.

- Beta Function: It is defined by:

$$B(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)}, \quad (92)$$

where Γ is the Gamma function given by expression (91). We shall notice that the above expression is not defined if $x, y \in \mathbb{Z}_-$

- Whittaker Functions $W_{\lambda,\mu}(z)$: The general case is given by expressions 9.220 of reference [Gradshteyn and Ryzhik 1981]. In this work, we apply a special kind obtained by setting $\lambda = 0$. In this case, we have:

$$W_{0,\mu}(z) = \sqrt{\frac{z}{\pi}} K_{\mu}\left(\frac{z}{2}\right), \quad (93)$$

where K_{μ} are the modified Bessel functions (equation 8.407, reference [Gradshteyn and Ryzhik 1981]):

$$K_{\mu}\left(\frac{z}{2}\right) = \frac{i\pi}{2} \exp\left(\frac{i\pi\mu}{2}\right) H_{\mu}^1\left(i\frac{z}{2}\right), \quad (94)$$

and (equations 8.405 and 8.403, reference [Gradshteyn and Ryzhik 1981]):

$$H_{\mu}^1\left(i\frac{z}{2}\right) = \mathbb{J}_{\mu}\left(i\frac{z}{2}\right) + i \frac{[\mathbb{J}_{\mu}\left(i\frac{z}{2}\right) \cos(\pi\mu) - \mathbb{J}_{-\mu}\left(i\frac{z}{2}\right)]}{\sin(\pi\mu)}, \quad (95)$$

with \mathbb{J}_{μ} being the Bessel functions (see bellow). These expressions can be computed only if: $\mu \notin \mathbb{Z}_{-}$ and $|\arg(z)| < \pi$.

- Bessel Functions: They are defined by [Filho 1986]:

$$\mathbb{J}_{\nu}(z) = \left(\frac{z}{2}\right)^{\nu} \sum_{k=0}^{+\infty} \frac{\left(-\frac{z^2}{4}\right)^k}{k! \Gamma(\nu + k + 1)}, \quad (96)$$

where ν is real and z can be complex. Considering the restriction for gamma function, we should not have $(\nu + k + 1) \in \mathbb{Z}_{-}$.

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